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On the universality of hadronisation corrections to QCD jets

Mrinal Dasgupta^a and Yazid Delenda^b

^a*School of Physics and Astronomy, University of Manchester,
Oxford road, Manchester M13 9PL, U.K.*

^b*Département de Physique, Faculté des Sciences, Université de Batna,
Batna, Algeria*

E-mail: Mrinal.Dasgupta@manchester.ac.uk, yazid.delenda@yahoo.com

ABSTRACT: We improve previously derived analytical estimates of hadronisation corrections to QCD jets at hadron colliders, firmly establishing at the two-loop level the link to the well-known power corrections to LEP event-shape variables. The results of this paper apply to jets defined in the k_t and anti- k_t algorithms but the general framework presented here holds also for other algorithms for which calculations are in progress.

KEYWORDS: Jets, QCD

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1 Introduction

The study of processes involving the production of high transverse momentum (p_t) QCD jets will form an integral part of the LHC physics program. In this light it is clear that the understanding of properties of jets will impact the accuracy of studies concerning new physics as well as traditional QCD studies concerning for instance the extraction of parton distribution functions.

While the subject of jet physics is far from new and has seen steady progress since the pioneering work of Sterman and Weinberg [1], the LHC provides a challenge of unprecedented complexity owing to the initial state hadronic environment and the very high

beam energy. The QCD tools at our disposal have therefore to be developed and adapted to meet this challenge.

The most important tool in the context of accurate jet studies at the LHC will remain QCD perturbation theory. Reliable perturbative estimates will constitute the only first principles (model-independent) information available from QCD theory and hence accurate perturbative results for various processes of interest are imperative. The advent of new infrared and collinear (IRC)-safe jet algorithms such as SISCone [2] and the anti- k_t [3] jet algorithms to complement existing IRC-safe algorithms such as the k_t [4, 5] and the Cambridge/Aachen algorithms [6] is therefore an encouraging and crucial development.

Perturbative aspects apart, another serious stumbling block for jet studies at the LHC is the role of non-perturbative effects. Here one is dealing not just with the hadronisation of partons but also the underlying event arising from beam remnant interactions, multiple hard interactions and the issue of pile-up. Thus while in an ideal world one may hope to construct clean signals for new physics, for instance mass peaks signifying the production of new particles, in the real hadron collider environment such peaks can be significantly smeared by non-perturbative QCD effects such as hadronisation and the underlying event, in addition to perturbative radiation. A better understanding of the size and role of non-perturbative effects as a function of the parameters used in jet studies (such as jet size) is hence imperative.

While neither hadronisation nor the underlying event can be calculated within perturbation theory one could argue that one may have more control on the former effect. The reason for this argument is that the hadronisation of partons that constitute a high- p_t jet is a phenomenon that has been met and handled before, for instance at LEP, HERA and the Tevatron. There has therefore been considerable development of hadronisation models in Monte Carlo event generators such as PYTHIA [7] and HERWIG [8].

Also significantly, in addition to Monte Carlo simulations, analytical techniques based on renormalons [9] have yielded important information on hadronisation corrections in the context of event-shape variables at LEP and HERA [10]. The pioneering phenomenological studies in this regard were carried out by Dokshitzer and Webber [11]. Using a model for a universal infrared-finite coupling they were able to compute the $1/Q$ power corrections to event-shape variables at LEP. The ensuing predictions involved just a single universal non-perturbative parameter, the moment of the coupling over the infrared region. Extracting the value of this moment from data on one variable it thus became possible to predict the $1/Q$ corrections to a large class of event-shape variables. Similar success was also met in the description of Breit-frame current-hemisphere event-shape variables at HERA [12] in spite of the presence of an initial-state proton that may potentially have impacted the power corrections.

The study of hadronisation corrections at hadron colliders, based on analytical techniques, is however still in its infancy. The main deterrents to such studies have been for instance the more complicated colour topology of multijet hard processes at hadron colliders and the competing presence of the underlying event. The issue of the role of colour in influencing the hadronisation corrections was explained in ref. [13] in the context of away-from-jet energy flow.

As far as the role of the underlying event compared to hadronisation is concerned an interesting result was derived in ref. [14]. There it was observed that if one studies the hadronisation contribution to the jet energy (specifically the change in p_t , δp_t of a jet due to hadronisation) using the aforementioned analytical techniques, one obtains a singular dependence¹ on the jet radius R with the hadronisation varying as $-1/R$. The underlying event on the other hand makes a contribution to the jet p_t varying as R^2 (see also refs. [16, 17]). The possibility thus emerges of a parametric separation of the two components of non-perturbative QCD with hadronisation being the dominant effect for jets at small R . Moreover the leading small- R behaviour was seen to arise due to *collinear* soft gluon radiation from the triggered hard parton jet. The colour factor associated to collinear radiation is merely the colour charge of the emitting hard parton, which means that in this limit the complex colour topology of the whole event is essentially irrelevant. Knowing the R -dependence of perturbative and non-perturbative corrections then allows for the possibility of deducing optimal R values adapted to different studies at hadron colliders [14].

Perhaps most interestingly a link was also made in ref. [14] between the magnitude of the $1/R$ hadronisation correction to the jet p_t and that of $1/Q$ power corrections to event shapes such as the thrust in e^+e^- annihilation [11]. The coefficient of the $1/R$ contribution is a perturbatively calculable number times a non-perturbative factor which is the same coupling moment as enters the event shape predictions. Carrying out the relevant perturbative calculation at the single-gluon level [14] the coefficient of the $1/R$ correction to the jet p_t for a quark jet is seen to be one-half of the coefficient of the $1/Q$ power correction for the e^+e^- thrust variable as computed in ref. [11].

The predictions of refs. [11] and [14] are based on one-loop $\mathcal{O}(\alpha_s)$ perturbative calculations. At this level the emission of a soft gluon with transverse momentum $k_t \sim \Lambda_{\text{QCD}}$ is associated to non-perturbative hadronisation contributions to the transverse momentum p_t of a hard jet or the event shape under consideration. Physically this is due to the fact that the running coupling associated to such an emission is $\alpha_s(k_t)$ which is perturbatively undefined at $k_t \sim \Lambda_{\text{QCD}}$. Replacing the unphysical perturbative coupling in this region by a finite and universal non-perturbative extension² leads to the predictions for hadronisation corrections reported in refs. [11] and [14].

In the context of event shapes however it was shown by Nason and Seymour that one-loop calculations involving single-gluon emission are inadequate to predict the coefficient of the $1/Q$ power corrections [19]. The argument of the running coupling which we took to be k_t can in fact only be reconstructed at the two-loop level where one has to account for the decay of the emitted gluon into offspring gluons or a quark/anti-quark pair. For variables such as event shapes and some observables involving jet definition such as jet p_t , one is *not* free to inclusively integrate over the decay products since the observable depends on the precise details of the decay kinematics. Thus in these cases the replacement of a gluon decay with a parent virtual (massive) gluon, with a running coupling dependent on the parent virtuality or k_t , is not actually correct at the level of power corrections. One

¹A similar result was obtained by Korchemsky and Sterman for cone jets in e^+e^- annihilation (see ref. [15]).

²The treatment of the running coupling was formalised via the dispersive approach adopted in ref. [18].

must then address the gluon decay and identify the correction needed to the single-gluon results quoted for example in refs. [11] and [14].

For the case of the thrust and other commonly studied e^+e^- event-shape variables calculations accounting for gluon decay have already been performed some time ago [20, 21] and yielded a simple result: taking into account gluon decay resulted in a universal (event-shape-independent) factor multiplying the single massive gluon results — the “Milan” factor. This was also seen to be the case for DIS event shapes [22] and further confirmed by the results of ref. [23]. One of the key ingredients in obtaining a universal Milan factor was the fact that all the event-shape variables considered are *linear* in transverse momenta of soft emitted particles [21]. The jet p_t variable is however not so simple. For almost all the commonly used jet algorithms it turns out that the δp_t due to gluon emission is not simply a linear sum over the contributions of individual gluons, and that the contribution of a given gluon depends on whether or not it is clustered or combined with other emissions. This effect is absent for the anti- k_t algorithm where soft gluons essentially cluster independently to the hard emitting parton and the result is the universal Milan factor [3].

In the present paper we establish that for jets defined in, for instance, the k_t algorithm (which we treat explicitly) the resulting rescaling factor is *not* the universal Milan factor. However we prove that it is a calculable factor of the same order, which we compute for the k_t algorithm. Thus it is now indeed possible to link the results for hadronisation corrections to jet p_t with those obtained for event shapes after including the Milan factor. As a result of our calculations the relation quoted in ref. [14] between the jet p_t hadronisation and the thrust power correction changes for all but the anti- k_t algorithm. The hadronisation corrections to jet p_t become algorithm-dependent rather than truly universal. They however remain universal in the more crucial sense that the result is always a perturbatively calculable number multiplying a universal non-perturbative coefficient. The only complication is that the perturbative calculations have to be at the two-loop level.

The current paper is organised as follows. In section 2 we deal with the impact of gluon emission and decay on the jet p_t considering both the kinematical effect on the observable as well as the dynamics of gluon branching. The naïve one-gluon term calculated in ref. [14] is identified and seen to be accompanied by two distinct correction terms, which following refs. [20, 21] we refer to as the inclusive and non-inclusive corrections. Section 3 deals with extracting the $1/R$ behaviour for the naïve result (where we recover the answer quoted in ref. [14]) and computing the inclusive correction. Section 4 is devoted to the treatment of the non-inclusive term for the k_t algorithm and the extraction of the leading $1/R$ behaviour in this term. In section 5 we combine the results to derive an overall rescaling factor to the result of ref. [14] for the k_t algorithm. We also recover the expected result for the anti- k_t algorithm which agrees with the Milan factor found for event shapes [20, 21]. Finally we draw some conclusions and discuss prospects for further work.

2 The jet δp_t and gluon splitting

Here we repeat the arguments, first detailed in ref. [20], that led to the emergence of the universal Milan factor, but do so in the context of the jet δp_t . We start with some introductory remarks that explain the simplifications we make which lead us to the final results.

2.1 Preliminary remarks

In ref. [14] it was noted that the dominant piece of the non-perturbative hadronisation contribution to the jet p_t at small jet radius R scaled as $1/R$. Additional terms $\mathcal{O}(R)$ and beyond were also obtained but these had a negligible impact at small³ R . Due to its significance we shall concentrate in this article on the dominant $1/R$ piece of the result.

It was also made clear in ref. [14] that the $1/R$ term arose from gluon emission at the boundary of the jet, which in the small- R limit is in the region collinear to the hard parton initiating the jet. To be precise the singular dependence on R is a direct manifestation of the collinear singularity present in the splitting of massless partons due to gluon emission. The same collinear singularity was also shown to be associated with $\ln R$ terms in the perturbative regime, which we do not discuss further here.

Given its origin in collinear radiation it is simple to deduce that the $1/R$ term does not depend on the details of the process of which the triggered hard parton (jet) is a part. This fact considerably simplifies the calculations involved and rather than considering all possible hard emitting dipoles in a given hard process (such as dijet production in hadron-hadron collisions treated in refs. [13, 14]) one need only consider collinear branching of the massless parton corresponding to the triggered hard jet.

In spite of this obvious generality, to illustrate the calculation and clarify the connection with the existing treatment [14], it may be useful to take a specific example. The example chosen in ref. [14] was the change in p_t due to hadronisation, δp_t , of a jet produced in the threshold limit of hadronic dijet production where threshold was defined by the limit $p_t \rightarrow \sqrt{s}/2$. Then the change in p_t from its threshold value due to soft gluon emission was computed and the gluon emission was averaged over to obtain the mean value of δp_t . In what follows we shall repeat the calculation of ref. [14], this time in the *collinear approximation* sufficient to generate the $1/R$ behaviour but with account of the decay of the emitted gluon.

2.2 Gluon emission kinematics and dynamics

We shall also be making use of the formulae and notation used in the Milan factor computations and first presented in ref. [20]. To this end let us introduce Sudakov vectors along and opposite to the triggered jet direction, taken itself to be at ninety degrees with respect to the beam as required by the threshold limit:

$$\begin{aligned} P &= \frac{\sqrt{s}}{2}(1, 1, 0, 0), \\ \bar{P} &= \frac{\sqrt{s}}{2}(1, -1, 0, 0). \end{aligned} \tag{2.1}$$

We parameterise the emitted parton four-momenta as in ref. [20]:

$$k_i = \alpha_i P + \beta_i \bar{P} + k_{ti}, \tag{2.2}$$

³As a matter of fact the $1/R$ behaviour was shown to hold to a good approximation over virtually the entire range of R values up to $R = 1$, which can in part be attributed to the smaller coefficients that emerged for the finite R pieces in the analysis of [14].

with \vec{k}_t the transverse momentum vector with respect to the jet direction. For massless partons we have that $k_i^2 = 0$ or $\alpha_i\beta_i = k_{ti}^2/s$. Also with neglect of recoil of the hard partons against the soft parton momenta we can identify P and \bar{P} with the momenta of the final-state hard partons themselves (recall that we are considering threshold where the hard partons are produced back-to-back and at ninety degrees to the beam).

We shall also need the value of δp_t due to soft parton emissions. Here one can make use of the results derived in ref. [14], where it was shown that emitted partons that are recombined with the triggered hard parton contribute to the δp_t by lending a mass M_j^2 to the jet. Soft partons not recombined with the jet lend a mass to the “recoil jet” involving the recoiling hard parton which is not triggered. Thus one has [14]:

$$\delta p_t = p_t - \frac{\sqrt{s}}{2} = - \left(\frac{M_j^2}{2\sqrt{s}} + \frac{M_r^2}{2\sqrt{s}} \right), \quad (2.3)$$

where the above equation is correct to first order in soft emitted parton momenta or equivalently the small jet masses. In terms of the Sudakov variables defined above one obtains:

$$\frac{M_j^2}{2\sqrt{s}} = \frac{1}{2\sqrt{s}} \left(p + \sum_{i \in j} k_i \right)^2 = \frac{\sqrt{s}}{2} \sum_{i \in j} \beta_i, \quad (2.4)$$

where the sum runs over all emissions recombined with the triggered hard parton, to form the triggered jet, and we neglected all terms quadratic in the transverse momenta of soft emitted partons including those that give a tiny recoil to the hard parton momentum p , which we thus identified with the Sudakov vector P . Likewise the mass of the recoil system is:

$$\frac{M_r^2}{2\sqrt{s}} = \frac{\sqrt{s}}{2} \sum_{i \notin j} \alpha_i, \quad (2.5)$$

where the sum is over all emissions not recombined into jet j . We shall use the above results in the following subsection.

2.3 Calculation of $\langle \delta p_t \rangle$ with gluon decay

Here we shall consider the change in p_t derived above, together with the squared matrix element for gluon production and decay, to obtain the average δp_t . We consider the situation up to the two-loop $\mathcal{O}(\alpha_s^2)$ level and hence we need to account for single gluon emission, virtual corrections to it as well as correlated two parton emission. To be precise we can write [20, 21]:

$$\langle \delta p_t \rangle = \frac{C_F}{\pi} \int \frac{d^2 k_t}{\pi k_t^2} \frac{d\alpha}{\alpha} \{ \alpha_s(0) + 4\pi\chi(k_t^2) \} \delta p_t(k) + 4C_F \int \left(\frac{\alpha_s}{4\pi} \right)^2 d\Gamma_2 \frac{M^2}{2!} \delta p_t(k_1, k_2), \quad (2.6)$$

where the first term on the right-hand side of the above represents the contribution from real massless gluon emission and the virtual correction to single emission denoted by $\chi(k_t^2)$. The second term on the right-hand side is the contribution from correlated two-parton emission with the squared matrix element for gluon decay M^2 containing both gluon decay into a

pair of gluons as well as the Abelian contribution from gluon branching to a $q\bar{q}$ pair. The measure $d\Gamma_2$ is the decay phase-space and $\delta p_t(k_1, k_2)$ is merely the change in p_t induced by two-parton emission, a special case of the general situation discussed above.

We should also mention the fact that the above result is in fact a “dressed” two-loop result where an infinite set of higher order perturbative graphs are implicitly embedded in the gluon propagator. These graphs are, at least in the Abelian limit, essentially self-energy or bubble insertions (renormalon graphs — see [9] for a review) that drive the scale of the coupling to the gluon virtuality k^2 , hence triggering non-perturbative contributions from the infrared region $k^2 \sim \Lambda_{\text{QCD}}^2$. Since for a real gluon the virtuality k^2 is zero, one has the ill-defined quantity $\alpha_s(0)$ appearing above (which will eventually disappear), while in the gluon decay case the argument of α_s is $k^2 = m^2$.

Note also that in writing eq. (2.6) we have specialised to the collinear limit by ignoring the fact that the soft gluon k can be emitted by several dipoles that form the hard particle ensemble, since in the collinear limit the emission is essentially off the triggered hard parton alone. We have taken this parton to be a quark and hence the appearance above of the colour factor C_F , but for a gluon jet this can straightforwardly be replaced by C_A .

We now follow all the arguments of refs. [20, 21]. We first attempt to treat δp_t as an inclusive quantity which corresponds to the treatment in ref. [14]. We then identify that there is a non-vanishing correction needed to the inclusive treatment which we shall compute. To this end we write:

$$\delta p_t(k_1, k_2) = \delta p_t(k_1, k_2) - \delta p_t(k_1 + k_2) + \delta p_t(k_1 + k_2), \tag{2.7}$$

where the term $\delta p_t(k_1 + k_2)$ corresponds to an inclusive treatment where the true value of $\delta p_t(k_1, k_2)$ is replaced by the value of δp_t that would be obtained by considering just the emission of the massive parent gluon $k_1 + k_2$, with $(k_1 + k_2)^2 = m^2$. The remainder $\delta p_t(k_1, k_2) - \delta p_t(k_1 + k_2)$ leads to a correction term that we evaluate in due course.⁴

Substituting (2.7) into (2.6) we can write:

$$\begin{aligned} \langle \delta p_t \rangle &= \frac{C_F}{\pi} \int \frac{d^2 k_t}{\pi k_t^2} \frac{d\alpha}{\alpha} \{ \alpha_s(0) + 4\pi\chi(k_t^2) \} \delta p_t(k) + \\ &+ 4C_F \int \left(\frac{\alpha_s}{4\pi} \right)^2 d\Gamma_2 \frac{M^2}{2!} \delta p_t(k_1 + k_2) + \langle \delta p_t \rangle^{ni}, \end{aligned} \tag{2.8}$$

where we separated the non-inclusive correction (denoted ni) from the inclusive approximation such that:

$$\langle \delta p_t \rangle^{ni} = 4C_F \int \left(\frac{\alpha_s}{4\pi} \right)^2 d\Gamma_2 \frac{M^2}{2!} (\delta p_t(k_1, k_2) - \delta p_t(k_1 + k_2)). \tag{2.9}$$

⁴It should be easy to see that this correction term is in general non-zero. If one considers for example a gluon decay where only one of the decay products say k_1 gets recombined with the jet, then one has $\delta p_t(k_1, k_2) = -\sqrt{s}/2(\alpha_2 + \beta_1)$. Assuming that the parent gluon is outside the jet the inclusive contribution is $\delta p_t(k_1 + k_2) = -\sqrt{s}/2\alpha = -\sqrt{s}/2(\alpha_1 + \alpha_2)$ which differs from $\delta p_t(k_1, k_2)$.

For the subsequent discussion we will need the results below for the two-parton (gluon decay) phase-space [20, 21]:

$$d\Gamma_2(k_1, k_2) = \prod_{i=1}^2 \frac{d\alpha_i}{\alpha_i} \frac{d^2 k_{ti}}{\pi} = \frac{d\alpha}{\alpha} \frac{d^2 k_t}{\pi} \frac{d^2 q}{\pi} z(1-z) dz, \quad (2.10)$$

with $\alpha = \alpha_1 + \alpha_2$ the Sudakov variable defined earlier, $\vec{k}_t = \vec{k}_{t1} + \vec{k}_{t2}$ the transverse momentum of the massive parent gluon, z the fraction α_1/α and q the relative “transverse angle” of the pair. Thus we have:

$$\alpha_1 = z\alpha, \quad \alpha_2 = (1-z)\alpha, \quad \vec{k}_t = \vec{k}_{t1} + \vec{k}_{t2}, \quad \vec{q} = \frac{\vec{k}_{t1}}{z} - \frac{\vec{k}_{t2}}{1-z}. \quad (2.11)$$

The above can also be straightforwardly expressed in terms of the mass of the parent gluon $m^2 = z(1-z)q^2$ so that:

$$d\Gamma_2(k_1, k_2) = \prod_{i=1}^2 \frac{d\alpha_i}{\alpha_i} \frac{d^2 k_{ti}}{\pi} = dm^2 \frac{d\alpha}{\alpha} \frac{d^2 k_t}{\pi} dz \frac{d\phi}{2\pi}, \quad (2.12)$$

where ϕ is the angle between \vec{k}_t and \vec{q} .

We note that the quantity M^2 is invariant under boosts in the longitudinal direction implying that it is independent of the variable α , which allows us to perform the α integral explicitly. Defining the result of the α integral by a *trigger function* Ω we can write the inclusive piece (first two terms on the right-hand side of eq. (2.8)) as:

$$\begin{aligned} \langle \delta p_t \rangle^i &= \frac{C_F}{\pi} \int \frac{d^2 k_t}{\pi k_t^2} \{ \alpha_s(0) + 4\pi\chi(k_t^2) \} \Omega_0(k_t^2) + \\ &+ 4C_F \int \left(\frac{\alpha_s}{4\pi} \right)^2 dm^2 \frac{d^2 k_t}{\pi} dz \frac{d\phi}{2\pi} \frac{M^2}{2!} \Omega_0(k_t^2 + m^2), \end{aligned} \quad (2.13)$$

with the trigger function being:

$$\Omega_0(k_t^2 + m^2) = \int_{(k_t^2+m^2)/s}^1 \frac{d\alpha}{\alpha} \delta p_t(\alpha, \beta = (k_t^2 + m^2)/(s\alpha)), \quad (2.14)$$

and similarly for $\Omega_0(k_t^2)$ which just corresponds to real massless gluon emission, and the superfix i pertains to the inclusive piece. In writing the above we have made use of the fact that δp_t depends on the α and β Sudakov variables and that $\beta = (k_t^2 + m^2)/(s\alpha)$ in the massive case and $k_t^2/(s\alpha)$ for massless real emission.

For the non-inclusive contribution a trigger function Ω_{ni} can be similarly defined so that it reads:

$$\langle \delta p_t \rangle^{ni} = 4C_F \int \left(\frac{\alpha_s}{4\pi} \right)^2 dm^2 \frac{d^2 k_t}{\pi} dz \frac{d\phi}{2\pi} \frac{M^2}{2!} \Omega_{ni}, \quad (2.15)$$

with

$$\Omega_{ni} = \int_{(k_t^2+m^2)/s}^1 \frac{d\alpha}{\alpha} (\delta p_t(k_1, k_2) - \delta p_t(k_1 + k_2)). \quad (2.16)$$

We shall first address the terms involving Ω_0 , the trigger functions for massive and massless single-gluon emission, and then focus on the non-inclusive term.

3 Dispersive calculations

Here we shall show how the leading non-perturbative terms may be extracted using the dispersive representation of the running coupling [18]. We start with the inclusive treatment, compute the relevant trigger functions and then move to the non-inclusive gluon decay term.

3.1 Naïve result and inclusive correction

Let us consider the terms in eq. (2.13) involving $\Omega_0(k_t^2)$ and $\Omega_0(k_t^2 + m^2)$. As was discussed in refs. [20, 21] the integral of the decay matrix element M^2 at fixed k_t gives:

$$\int \left(\frac{\alpha_s}{4\pi}\right)^2 dm^2 dz \frac{d\phi}{2\pi} \frac{M^2}{2!} = \int_0^\infty \frac{dm^2}{m^2(m^2 + k_t^2)} \left(\frac{\alpha_s}{4\pi}\right)^2 \left\{ -\beta_0 + 2C_A \ln \frac{k_t^2(k_t^2 + m^2)}{m^4} \right\}, \quad (3.1)$$

which contains a term proportional to β_0 as well as a singular term where the singularity is associated to the collinear gluon splitting. In writing the above we have, as in refs. [18, 20, 21], made use of the ultraviolet convergence of the integrals to extend the upper limits of the m^2 integration to infinity, since we are concerned here with the infrared region $k_t, m \sim \Lambda_{\text{QCD}}$. The collinear singularity cancels against a similar one in the function χ [20, 21]:

$$\frac{\chi(k_t^2)}{k_t^2} = \int_0^\infty \frac{d\mu^2}{\mu^2(\mu^2 + k_t^2)} \left(\frac{\alpha_s}{4\pi}\right)^2 \left\{ -2C_A \ln \frac{k_t^2(k_t^2 + \mu^2)}{\mu^4} \right\} + 2S \left(\frac{\alpha_s}{2\pi}\right)^2, \quad (3.2)$$

leaving a finite remainder. Here S is a scheme-dependent constant. Also the term proportional to β_0 combines with the $\alpha_s(0)$ term to build up the running coupling in the CMW scheme ($S = 0$) [24], which shall always be implied henceforth. To be more precise, after integrating over the decay phase-space $d\Gamma_2$, eq. (2.13) can be written as a sum of two finite terms:

$$\langle \delta p_t \rangle^i = \langle \delta p_t \rangle^0 + \langle \delta p_t \rangle^{\text{inc}}. \quad (3.3)$$

Let us first describe the term $\langle \delta p_t \rangle^0$ which reads [21]:

$$\begin{aligned} \langle \delta p_t \rangle^0 &= 4C_F \int \frac{dm^2 dk_t^2}{k_t^2 + m^2} \left\{ \frac{\alpha_s(0)}{4\pi} \delta(m^2) - \frac{\beta_0}{m^2} \left(\frac{\alpha_s}{4\pi}\right)^2 \right\} \Omega_0(k_t^2 + m^2) \\ &= \frac{C_F}{\pi} \int_0^\infty dm^2 \alpha_{\text{eff}}(m^2) \frac{-d}{dm^2} \int_0^{Q^2} \frac{dk_t^2}{k_t^2 + m^2} \Omega_0(k_t^2 + m^2) \\ &= \frac{C_F}{\pi} \int_0^\infty \frac{dm^2}{m^2} \alpha_{\text{eff}}(m^2) \Omega_0(m^2), \end{aligned} \quad (3.4)$$

where we wrote $Q^2 \approx s$ as the upper limit of the k_t integral, which is redundant here since our final results stem from the low- k_t infrared regime. The above result involves invoking the *effective coupling* which is related to α_s via a dispersive integral [18]:

$$\frac{\alpha_s(k^2)}{k^2} = \int_0^\infty dm^2 \frac{\alpha_{\text{eff}}(m^2)}{(m^2 + k^2)^2}, \quad (3.5)$$

so that one has:

$$\frac{d}{d \ln m^2} \frac{\alpha_{\text{eff}}(m^2)}{4\pi} = -\beta_0 \left(\frac{\alpha_s}{4\pi}\right)^2 + \dots, \quad \alpha_{\text{eff}}(0) = \alpha_s(0). \quad (3.6)$$

To arrive at the last line of eq. (3.4) we substituted the effective coupling instead of α_s using eq. (3.6) and performed an integration by parts. The term corresponding to $\langle \delta p_t \rangle^0$ is the one that is produced by ignoring all two-loop effects except those included via the running coupling. It thus corresponds to a naïve single-gluon treatment and is thus referred to as the *naïve* term. This is the term that will be related to the one-gluon emission study in ref. [14], where a running coupling $\alpha_s(k_t)$ was introduced by hand into the calculation.⁵

The term $\langle \delta p_t \rangle^{\text{inc}}$ is a leftover from the cancellation of singularities between the function χ and the collinear divergent piece of the result for the integral of M^2 [20, 21], which following refs. [20, 21] we refer to as the inclusive correction to the naïve result:

$$\begin{aligned} \langle \delta p_t \rangle^{\text{inc}} &= \frac{8C_F C_A}{\beta_0} \int \frac{dm^2}{m^2} \frac{\alpha_{\text{eff}}(m^2)}{4\pi} \frac{d}{d \ln m^2} \int \frac{dk_t^2}{k_t^2 + m^2} \ln \frac{k_t^2(k_t^2 + m^2)}{m^4} \Omega^{\text{inc}}(k_t^2 + m^2), \\ \Omega^{\text{inc}} &\equiv \Omega_0(k_t^2 + m^2) - \Omega_0(k_t^2). \end{aligned} \quad (3.7)$$

The inclusive correction merely arises out of the difference between the massive gluon and massless gluon trigger functions which we denote by Ω^{inc} , and is the first sign of two-loop effects not included simply via the use of a running coupling.

3.1.1 Naïve result

In order to focus on the non-perturbative contribution alone we can split $\alpha_s = \alpha_s^{\text{PT}} + \delta\alpha_s$, with $\delta\alpha_s$ a non-perturbative modification to the perturbative definition of α_s which results in the corresponding modification to the effective coupling $\delta\alpha_{\text{eff}}$.

We can also take the small- m^2 limit of the trigger function $\Omega_0(m^2)$ so that the non-perturbative (NP) contribution reads:

$$\langle \delta p_t \rangle_{\text{NP}}^0 = \frac{C_F}{\pi} \int_0^\infty \frac{dm^2}{m^2} \delta\alpha_{\text{eff}}(m^2) \Omega_0(m^2). \quad (3.8)$$

The non-perturbative modification $\delta\alpha$ must vanish in the perturbative region so that the coupling α_{eff} matches on to the perturbative coupling. Hence the integral above, representing the non-perturbative contribution, has support only over a limited range of m^2 up to some arbitrarily chosen perturbative matching scale μ_I . The μ_I -dependence of the result is then cancelled up to $\mathcal{O}(\alpha_s^n)$ when combining the non-perturbative result with a perturbative estimate at the same order. This issue, mentioned for the sake of completeness, is irrelevant to the rest of our discussion and shall henceforth not be mentioned again. We shall also avoid writing the subscript NP since we always refer to the non-perturbative contributions from now on.

⁵Strictly speaking the running coupling used in ref. [14] had as argument the transverse momentum with respect to the dipole axis in the rest frame of each emitting dipole. For generating the $1/R$ collinear term it suffices to replace this with k_t , the transverse momentum with respect to the emitting hard leg.

Let us compute the naïve trigger function $\Omega_0(k_t^2 + m^2)$ in order to evaluate eq. (3.8). This is given by:

$$\Omega_0(k_t^2 + m^2) = \int_{(k_t^2+m^2)/s}^1 \frac{d\alpha}{\alpha} \delta p_t(k), \quad (3.9)$$

where $k = k_1 + k_2$ is the parent (massive) gluon four-momentum. Using the results of the previous section eqs. (2.3) to (2.5) we have:

$$\begin{aligned} \delta p_t(k) &= \delta p_t^+ + \delta p_t^- \\ &= -\frac{\sqrt{s}}{2} \beta \Theta_{\text{in}}(k) - \frac{\sqrt{s}}{2} \alpha \Theta_{\text{out}}(k), \end{aligned} \quad (3.10)$$

where Θ_{out} and Θ_{in} are step functions indicating whether the gluon is outside or inside the jet (i.e. if it is recombined with the hard parton or not).

In all the jet algorithms that we consider, the clustering of a single particle (e.g. a soft gluon) to the triggered hard parton is based on the distance criterion $\delta\eta^2 + \delta\phi^2$, with $\delta\eta$ and $\delta\phi$ the separation in rapidity and azimuth between the particle considered and the hard parton. At small angles (i.e. in the collinear limit relevant to our results) this is just θ^2 , where θ is the small angle between the soft gluon and the triggered hard parton. Thus at the single gluon level the soft gluon is combined with the hard parton if $\theta^2 < R^2$, and is not combined if the converse is true. Moreover using the easily-derived relation⁶ $\theta^2 = 4\beta/\alpha$ we have:

$$\Theta_{\text{out}}(k) = \Theta(4\beta/\alpha - R^2) = \Theta\left(2\frac{\sqrt{k_t^2 + m^2}}{R\sqrt{s}} - \alpha\right), \quad (3.11)$$

where we used $\alpha\beta = (k_t^2 + m^2)/s$, and conversely for $\Theta_{\text{in}}(k)$.

Let us first consider the $\Theta_{\text{out}} = 1$ case or the gluon not recombined with the jet. The contribution to the trigger function is:

$$\Omega_0(k_t^2 + m^2) = -\frac{\sqrt{s}}{2} \int_{(k_t^2+m^2)/s}^1 \frac{d\alpha}{\alpha} \alpha \Theta\left(2\frac{\sqrt{k_t^2 + m^2}}{R\sqrt{s}} - \alpha\right). \quad (3.12)$$

Performing the above integral we arrive at:

$$\Omega_0(k_t^2 + m^2) = -\frac{\sqrt{k_t^2 + m^2}}{R}, \quad (3.13)$$

where we retained only the leading $\mathcal{O}(k_t/R)$ term we wish to compute, neglecting terms of order k_t^2 that arise from the lower limit of the integral. Thus the $1/R$ behaviour is associated to quasi-collinear soft emission such that $1 \gg \alpha \gg \beta$, where $1 \gg \alpha$ corresponds to soft emission, while $\alpha \gg \beta$ is the collinear regime. We note that the convergence of the α integral can be used to extend the integral to infinity, while the irrelevant lower limit can be set to zero as in refs. [20, 21], $\int d\alpha/\alpha \rightarrow \int_0^\infty d\alpha/\alpha$. Lastly we just mention that the term

⁶Strictly speaking this relation is true for massless gluon clustering. In the naïve calculation we are however free to ignore the gluon mass in the definition of the clustering angle, since precisely the same quantity is subtracted in the inclusive and non-inclusive terms at the end. This freedom to define the role of mass in the naïve calculation was also exploited in refs. [20, 21] to obtain a trigger function $\propto \sqrt{k_t^2 + m^2}$.

with the soft gluon recombined with the jet, δp_t^+ , does not contribute to the $1/R$ result. The physical reason for this is straightforward: the contribution to the jet δp_t from this term is proportional to β or the jet mass, which vanishes in the collinear limit responsible for the $1/R$ behaviour, i.e. is collinear-safe. Thus only gluons not recombined to the hard jet give us a hadronisation correction varying as $1/R$.

Now we substitute eq. (3.13) into the result eq. (3.8) to obtain the naïve result:

$$\langle \delta p_t \rangle^0 = -\frac{1}{R} \frac{C_F}{\pi} \int_0^\infty \frac{dm^2}{m^2} \delta\alpha_{\text{eff}}(m^2) m. \quad (3.14)$$

This result can be written as:

$$\langle \delta p_t \rangle^0 = -\frac{2A_1}{R}, \quad (3.15)$$

where we introduced the non-perturbative coupling moment A_1 :

$$A_1 = \frac{C_F}{2\pi} \int_0^\infty \frac{dm^2}{m^2} m \delta\alpha_{\text{eff}}(m^2). \quad (3.16)$$

The coefficient $2A_1$ of the $-1/R$ term can be compared directly to the naïve result for e^+e^- event-shape variables [20, 21]. It is precisely one-half of the result for the thrust mean-value exactly as was reported in [14] also within a naïve single-gluon approach.

3.1.2 Inclusive correction

Next we deal with the inclusive correction term. Here the trigger function is just the difference between the massive and massless gluon phase-space:

$$\begin{aligned} \Omega^{\text{inc}} &= -\frac{\sqrt{s}}{2} \int_0^\infty \frac{d\alpha}{\alpha} \Theta \left(2\frac{\sqrt{k_t^2 + m^2}}{R\sqrt{s}} - \alpha \right) + \frac{\sqrt{s}}{2} \int_0^\infty \frac{d\alpha}{\alpha} \Theta \left(\frac{2k_t}{R\sqrt{s}} - \alpha \right) \\ &= -\frac{1}{R} \left(\sqrt{k_t^2 + m^2} - k_t \right). \end{aligned} \quad (3.17)$$

The inclusive trigger function is precisely of the form one meets in refs. [20, 21], with the coefficient of $-1/R$, being identical to the corresponding quantity for event-shape variables. Inserting the trigger function into eq. (3.7) and introducing $\delta\alpha_{\text{eff}}(m^2)$ as before, one gets:

$$\langle \delta p_t \rangle^{\text{inc}} = \langle \delta p_t \rangle^0 r_{in}, \quad (3.18)$$

with

$$r_{in} = \frac{2C_A}{\beta_0} \int_0^\infty \frac{xdx}{1+x^2} \frac{\ln[x^2(x^2+1)]}{x+\sqrt{1+x^2}} \approx 3.299 \frac{C_A}{\beta_0}, \quad (3.19)$$

where we defined $x = k_t/m$ as in refs. [20, 21]. The factor r_{in} is the same as for the thrust and other event shapes so adding the inclusive correction to the naïve term, the result for the jet δp_t is still one-half of the corresponding result for the thrust, i.e. universality is maintained at the inclusive level.

Having computed the naïve and inclusive pieces we are left with the non-inclusive correction. It is here that the jet δp_t result will no longer receive the same correction as event shapes and universality of the Milan factor will be broken (except for the case of the anti- k_t jet algorithm).

3.2 Non-inclusive correction

For the non-inclusive term one can follow a similar treatment to the inclusive case but with the appropriate non-inclusive trigger function. Then eq. (2.15) becomes, as for the case of event-shape variables [20, 21],

$$\langle \delta p_t \rangle^{ni} = 4C_F \int_0^\infty \frac{dm^2}{m^2} \frac{\delta\alpha_{\text{eff}}(m^2)}{4\pi} \frac{d}{d \ln m^2} \left\{ \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^1 dz \int dk_t^2 m^2 \frac{1}{2!} M^2 \frac{1}{\beta_0} \Omega_{ni} \right\}, \quad (3.20)$$

where we used eq. (3.6) and integrated (2.15) by parts. We shall focus our attention on the term in parenthesis that yields the linear behaviour in m corresponding to the leading non-perturbative (hadronisation) correction. In order to compute this we need to evaluate the non-inclusive trigger function:

$$\int_0^\infty \frac{d\alpha}{\alpha} (\delta p_t(k_1, k_2) - \delta p_t(k_1 + k_2)). \quad (3.21)$$

The term $\delta p_t(k_1, k_2)$ represents the contribution from two-parton emission, which depends on the details of whether the offspring partons end up inside or outside the jet *after* the application of the jet algorithm. For most jet algorithms currently in use, this involves more than just working out whether the gluons are within a distance R of the hard parton (an exception is the anti- k_t algorithm), so we shall need to consider the action of the algorithms in some detail. Due to the length of this discussion we shall devote the next section entirely to it.

4 The non-inclusive trigger function

We shall start by considering the k_t algorithm. Let us briefly remind the reader of how this works.

First one computes the distances d_{ij} between all pairs of objects i and j , $d_{ij} = \min(\kappa_{t,i}^2, \kappa_{t,j}^2) y_{ij}$, with $y_{ij} = \delta\eta^2 + \delta\phi^2$, where $\delta\eta$ and $\delta\phi$ refer to the separation of i and j in rapidity and azimuth measured with respect to the beam, and we used κ_t to denote the transverse momentum with respect to the beam direction (while reserving k_t for the transverse momentum with respect to the hard emitting parton). In the small angle approximation appropriate to our work we just have $\delta\eta^2 + \delta\phi^2 \rightarrow \theta_{ij}^2$, where θ_{ij} is the angle between i and j . One also computes the distances d_{iB} between each object and the beam defined as $\kappa_{t,i}^2 R^2$, with R the selected radius parameter.

Then if amongst the various distances the smallest is a d_{iB} , object i is a jet and is removed from the list of objects to be clustered. If the smallest distance is a d_{ij} then i and j are combined (merged) into a single object and the procedure is then iterated until all objects have been removed. The recombination scheme we use here is addition of four-momenta so that the object resulting from combining i and j has four-momentum $p_i + p_j$.

We note from eqs. (2.3) to (2.5) that the $\delta p_t(k_1, k_2)$ is additive over the contributions of k_1 and k_2 ,

$$\begin{aligned} \delta p_t(k_1, k_2) &= \delta p_t(k_1) + \delta p_t(k_2) \\ &= \delta p_t^+(k_1) \Xi_{\text{in}}(k_1) + \delta p_t^-(k_1) \Xi_{\text{out}}(k_1) + (k_1 \leftrightarrow k_2), \end{aligned} \quad (4.1)$$

Θ_{12}	Θ_{j1}	Θ_{j2}	$\delta p_t(k_1, k_2)$
1	1	1	$\delta p_t^{++}(k_1, k_2)$
1	1	0	Discussed below
1	0	1	Discussed below
1	0	0	Discussed below
0	1	1	$\delta p_t^{+-}(k_1, k_2)$
0	1	0	$\delta p_t^{+0}(k_1, k_2)$
0	0	1	$\delta p_t^{-+}(k_1, k_2)$
0	0	0	$\delta p_t^{--}(k_1, k_2)$

Table 1. Values of δp_t depending on the angular separations of partons. The three entries “Discussed below” need special treatment and thus are left for discussion in the text.

where we expressed δp_t in terms of the contributions from k_1 and k_2 . We also distinguished the cases when the offspring partons are recombined into and fly outside the jets as the contributions are different in either case:

$$\begin{aligned}\delta p_t^+(k_i) &= -\frac{\sqrt{s}}{2}\beta_i, \\ \delta p_t^-(k_i) &= -\frac{\sqrt{s}}{2}\alpha_i,\end{aligned}\tag{4.2}$$

where δp_t^+ expresses the contribution from a recombined parton and δp_t^- for a non-recombined emission. The conditions $\Xi_{\text{in,out}}$ denote the constraints for partons to be in and outside the jets and they are no longer simple step functions as for the single-gluon case, and must be obtained by applying the algorithm in full including the possibility of soft partons clustering to each other.

4.1 δp_t in the k_t algorithm

Let us first apply the k_t algorithm to work out the $\delta p_t(k_1, k_2)$ in various situations. We have to consider the soft partons k_1, k_2 and their distances from the triggered hard parton “jet” which we denote by j . To this end let us define the step functions $\Theta_{ab} = \Theta(R^2 - \theta_{ab}^2)$, so that if two partons have an angular separation less than R the step function is unity and else it is zero.⁷ Then we can divide the full phase-space up into definite regions which account for all possible cases for the angular separations, as indicated by table 1. For each row of the table, i.e. each of the possible combinations of values of the Θ_{ab} , it is possible to obtain a value for the $\delta p_t(k_1, k_2)$.

In table 1 we have written $\delta p_t^{++}(k_1, k_2) = \delta p_t^+(k_1) + \delta p_t^+(k_2)$ to indicate the contribution when both k_1 and k_2 are eventually combined into the jet, while $\delta p_t^{+-}(k_1, k_2) = \delta p_t^+(k_1) + \delta p_t^-(k_2)$ indicates a situation where k_1 is recombined and k_2 is not. Let us show how the entries in this table are obtained by considering an example and discussing the more interesting scenarios.

⁷Recall that we are looking at the small- R behaviour $R \ll 1$, which entitles us to replace $\delta\eta^2 + \delta\phi^2$ by the small angle approximation θ^2 . Neglected terms produce only finite R corrections.

The scenario: $\Theta_{12} = 1, \Theta_{j1} = 1, \Theta_{j2} = 0$. We consider the case described in the second row of the above table. The distances we need to discuss are the following:

$$\begin{aligned} d_{1B} &= \kappa_{t,1}^2 R^2, \\ d_{2B} &= \kappa_{t,2}^2 R^2, \\ d_{12} &= \min(\kappa_{t,1}^2, \kappa_{t,2}^2) \theta_{12}^2, \\ d_{1j} &= \kappa_{t,1}^2 \theta_{1j}^2, \\ d_{2j} &= \kappa_{t,2}^2 \theta_{2j}^2. \end{aligned}$$

In this situation (given the values of the Θ functions) it is not possible for any of d_{1B} , d_{2B} , or d_{2j} to be the smallest. Thus either d_{12} or d_{1j} is the smallest.

If d_{12} is the smallest (i.e. smaller than d_{1j}) then k_1 and k_2 are clustered to each other first. Two sub-scenarios arise: the resultant soft jet may fall either within or outside the grasp of the hard parton. If the soft jet is recombined with the hard parton the resulting contribution is δp_t^{++} , else it is δp_t^{--} .

Alternatively if d_{1j} is the smallest distance particle k_1 is combined with the jet first. Then particle k_2 forms a separate jet, i.e. is removed as it is closer to the beam than it is to the hard jet. The contribution is then $\delta p_t^{+-}(k_1, k_2)$. We can summarise the overall scenario just discussed by the result:

Case $d_{12} < d_{1j}$:

$$\delta p_t(k_1, k_2) = \Theta(d_{1j} - d_{12}) [\Theta(R^2 - \theta_{kj}^2) \delta p_t^{++} + \Theta(-R^2 + \theta_{kj}^2) \delta p_t^{--}], \quad (4.3)$$

where θ_{kj} is the angle of the soft jet (parent gluon) $k = k_1 + k_2$ with the hard parton.

Case $d_{12} > d_{1j}$: In this case particle 1 gets clustered to the jet and 2 is left outside. Thus:

$$\delta p_t(k_1, k_2) = \Theta(d_{12} - d_{1j}) \delta p_t^{+-}(k_1, k_2). \quad (4.4)$$

The scenario: $\Theta_{12} = 1, \Theta_{j1} = 0, \Theta_{j2} = 1$. This scenario (third row of the table above) is the same as the one just discussed but with k_1 and k_2 interchanged.

The scenario: $\Theta_{12} = 1, \Theta_{j1} = 0, \Theta_{j2} = 0$. Here the smallest quantity is d_{12} and the soft partons are clustered to each other first. The resulting soft jet may or may not be recombined with the hard parton. Thus we have in this region:

$$\delta p_t(k_1, k_2) = \Theta(R^2 - \theta_{kj}^2) \delta p_t^{++} + \Theta(-R^2 + \theta_{kj}^2) \delta p_t^{--}, \quad (4.5)$$

where the step functions specify whether the soft parton jet is within an angular distance R of the hard parton or not.

All the other scenarios follow straightforwardly and the values of the δp_t are as indicated in the table. For instance if one considers the bottom row of the table where all the

Θ functions vanish (partons always separated by more than R) the d_{iB} are always smallest, and the partons k_1 and k_2 are never combined into the triggered jet, so the result is δp_t^{--} .

Now we are in position to write down the expression for δp_t at two-parton level. We express it as follows:

$$\begin{aligned}
 \delta p_t(k_1, k_2) = & \delta p_t^{++}(k_1, k_2)\Theta_{\text{in}}(k_1)\Theta_{\text{in}}(k_2) + \\
 & + \delta p_t^{--}(k_1, k_2)\Theta_{\text{out}}(k_1)\Theta_{\text{out}}(k_2) [1 - \Theta_{12}(k_1, k_2) + \Theta_{12}(k_1, k_2)\Theta_{\text{out}}(k)] + \\
 & + \delta p_t^{++}(k_1, k_2)\Theta_{\text{out}}(k_1)\Theta_{\text{out}}(k_2)\Theta_{12}(k_1, k_2)\Theta_{\text{in}}(k) + \\
 & + \delta p_t^{++}(k_1, k_2)\Theta_{\text{in}}(k_1)\Theta_{\text{out}}(k_2)\Theta_{12}(k_1, k_2)\Theta(d_{1j} - d_{12})\Theta_{\text{in}}(k) + (k_1 \leftrightarrow k_2) + \\
 & + \delta p_t^{--}(k_1, k_2)\Theta_{\text{in}}(k_1)\Theta_{\text{out}}(k_2)\Theta_{12}(k_1, k_2)\Theta(d_{1j} - d_{12})\Theta_{\text{out}}(k) + (k_1 \leftrightarrow k_2) + \\
 & + \delta p_t^{+-}(k_1, k_2)\Theta_{\text{in}}(k_1)\Theta_{\text{out}}(k_2)\Theta_{12}(k_1, k_2)\Theta(d_{12} - d_{1j}) + (k_1 \leftrightarrow k_2) + \\
 & + \delta p_t^{+-}(k_1, k_2)\Theta_{\text{in}}(k_1)\Theta_{\text{out}}(k_2)(1 - \Theta_{12}) + (k_1 \leftrightarrow k_2). \tag{4.6}
 \end{aligned}$$

In the above $\Theta_{\text{in}}(k_i) = \Theta(R^2 - \theta_{ij}^2)$ and $\Theta_{\text{out}} = 1 - \Theta_{\text{in}}$ pertain to whether the parton k_i is within or outside an angular separation R of the hard parton, while $\Theta_{12} = \Theta(R^2 - \theta_{12}^2)$ is the constraint for the soft partons to have an angular separation less than R . Eq. (4.6) merely summarises the contents of the table, for instance the sum of entries in the first and fifth rows is just the first line of the above equation. For consistency one can check that by removing all the δp_t terms in the above equation and adding all the theta functions as they appear one should get 1. The check is positive (meaning all possible scenarios have been discussed without double counting).

We can also write eq. (4.6) in terms of the individual contributions of k_1 and k_2 since the δp_t is additive over the contributions of individual partons:

$$\delta p_t(k_1, k_2) = \delta p_t(k_1) + \delta p_t(k_2) = \delta p_t^-(k_1)\Xi_{\text{out}}(k_1) + \delta p_t^+(k_1)\Xi_{\text{in}}(k_1) + (k_1 \leftrightarrow k_2), \tag{4.7}$$

where we separated the contributions from each offspring parton into its + and - components according to whether the parton is eventually recombined with the jet or not.

Next we note that for the $1/R$ term one is only interested in partons not recombined to the jet. Partons recombined with the jet contribute to the δp_t via $\delta p_t^+(k_i) \propto \beta_i$. Integrating such contributions over the $d\alpha/\alpha$ phase-space produces only terms regular in R , as for the single-gluon case. Thus for computing the $1/R$ term the $\delta p_t(k_1, k_2)$ can just be expressed as:

$$\delta p_t(k_1, k_2) = \delta p_t^-(k_1)\Xi_{\text{out}}(k_1) + \delta p_t^-(k_2)\Xi_{\text{out}}(k_2). \tag{4.8}$$

Thus from eq. (4.6), focussing on all terms where k_1 is not in the jet we can identify:

$$\begin{aligned}
 \Xi_{\text{out}}(k_1) = & \Theta_{\text{out}}(k_1)\Theta_{\text{out}}(k_2) [1 - \Theta_{12}(k_1, k_2) + \Theta_{12}(k_1, k_2)\Theta_{\text{out}}(k)] + \\
 & + \Theta_{\text{in}}(k_1)\Theta_{\text{out}}(k_2)\Theta_{12}(k_1, k_2)\Theta(d_{1j} - d_{12})\Theta_{\text{out}}(k) + \\
 & + \Theta_{\text{in}}(k_2)\Theta_{\text{out}}(k_1)\Theta_{12}(k_1, k_2)\Theta(d_{2j} - d_{12})\Theta_{\text{out}}(k) + \\
 & + \Theta_{\text{in}}(k_2)\Theta_{\text{out}}(k_1)\Theta_{12}(k_1, k_2)\Theta(d_{12} - d_{2j}) + \\
 & + \Theta_{\text{in}}(k_2)\Theta_{\text{out}}(k_1)(1 - \Theta_{12}), \tag{4.9}
 \end{aligned}$$

which can be simplified after some algebra to:

$$\begin{aligned}\Xi_{\text{out}}(k_1) &= \Theta_{\text{out}}(k_1) [1 - \Theta_{\text{out}}(k_2)\Theta_{12}(k_1, k_2)\Theta_{\text{in}}(k)] + \\ &\quad + \Theta_{\text{in}}(k_1)\Theta_{\text{out}}(k_2)\Theta_{12}(k_1, k_2)\Theta(d_{1j} - d_{12})\Theta_{\text{out}}(k) + \\ &\quad - \Theta_{\text{out}}(k_1)\Theta_{\text{in}}(k_2)\Theta_{12}(k_1, k_2)\Theta(d_{2j} - d_{12})\Theta_{\text{in}}(k),\end{aligned}\quad (4.10)$$

which represents all possible ways for k_1 to be outside the jet *after* application of the k_t algorithm. A similar expression holds for k_2 .

Repeating the above procedure for the anti- k_t algorithm yields a much simpler result due to the absence of soft parton self-clustering. Thus in the anti- k_t case a given parton k_i is always outside the jet if it is separated by more than R in angle from the hard parton:

$$\Xi_{\text{out}}(k_i) = \Theta_{\text{out}}(k_i).\quad (4.11)$$

4.2 The α integral

Having worked out the relevant piece of the $\delta p_t(k_1, k_2)$ for the k_t algorithm we now proceed to carry out the integral over α involved in the definition of the trigger function eq. (2.16). This is given by:

$$\begin{aligned}\Omega_{ni} &= \int \frac{d\alpha}{\alpha} (\delta p_t^-(k_1)\Xi_{\text{out}}(k_1) + \delta p_t^-(k_2)\Xi_{\text{out}}(k_2) - \delta p_t^-(k)\Theta_{\text{out}}(k)) \\ &= -\frac{\sqrt{s}}{2} \int \frac{d\alpha}{\alpha} (\alpha_1\Xi_{\text{out}}(k_1) + \alpha_2\Xi_{\text{out}}(k_2) - \alpha\Theta_{\text{out}}(k)) \\ &= -\frac{\sqrt{s}}{2} \int d\alpha (z\Xi_{\text{out}}(k_1) + (1-z)\Xi_{\text{out}}(k_2) - \Theta_{\text{out}}(k)) \\ &= \Omega_1 + \Omega_2 - \Omega_3,\end{aligned}\quad (4.12)$$

where

$$\begin{aligned}\Omega_1 &= -\frac{\sqrt{s}}{2} z \int_0^\infty d\alpha \left\{ \Theta_{\text{out}}(k_1) [1 - \Theta_{\text{out}}(k_2)\Theta_{12}(k_1, k_2)\Theta_{\text{in}}(k)] + \right. \\ &\quad \left. + \Theta_{\text{in}}(k_1)\Theta_{\text{out}}(k_2)\Theta_{12}(k_1, k_2)\Theta(d_{1j} - d_{12})\Theta_{\text{out}}(k) + \right. \\ &\quad \left. - \Theta_{\text{out}}(k_1)\Theta_{\text{in}}(k_2)\Theta_{12}(k_1, k_2)\Theta(d_{2j} - d_{12})\Theta_{\text{in}}(k) \right\},\end{aligned}\quad (4.13)$$

with e.g. $\Theta_{\text{out}}(k_1) = \Theta(\theta_{1j}^2 - R^2)$. Note that in writing the above we have used $\delta p_t^-(k_i) = -\alpha_i\sqrt{s}/2$, with $\alpha_1 = z\alpha$ and $\alpha_2 = (1-z)\alpha$. We shall give expressions for Ω_2 and Ω_3 in due course.

In order to proceed we shall also need to express the angles between partons in terms of the Sudakov variables. To this end we define:

$$\vec{q}_1 = \frac{\vec{k}_{t1}}{z}, \quad \vec{q}_2 = \frac{\vec{k}_{t2}}{1-z}, \quad \vec{q} = \vec{q}_1 - \vec{q}_2.\quad (4.14)$$

The angle between the soft parton k_1 and the hard parton j is just (assuming $\alpha_i \gg \beta_i$, which is the collinear approximation) $\theta_{1j}^2 = 4\beta_1/\alpha_1 = 4k_{t1}^2/(s\alpha_1^2)$ and similarly we have $\theta_{2j}^2 = 4k_{t2}^2/(s\alpha_2^2)$. Moreover since $\alpha_1 = z\alpha$ and $\alpha_2 = (1-z)\alpha$ we have:

$$\theta_{1j}^2 = \frac{4q_1^2}{s\alpha^2}, \quad \theta_{2j}^2 = \frac{4q_2^2}{s\alpha^2}.\quad (4.15)$$

The angle between the soft jet k , formed by recombining k_1 and k_2 , and the hard parton j is given (again in the collinear limit) by:

$$\theta_{kj}^2 = \frac{4k_t^2}{s\alpha^2}, \quad (4.16)$$

and finally the angle between k_1 and k_2 is just:

$$\theta_{12}^2 = \frac{4q^2}{s\alpha^2}. \quad (4.17)$$

In writing these angles we have used the small angle approximation throughout, specifically $(1 - \cos \theta) \approx \theta^2/2$, and also considered $\alpha_i \gg \beta_i$ as we are always concerned with partons in the region collinear to the emitting hard parton. Terms neglected in this approximation contribute to corrections regular in R .

Next we rescale $\sqrt{s}R\alpha/2 \rightarrow \alpha$ so as to straightforwardly extract the $1/R$ -dependence that concerns us. Hence we write:

$$\begin{aligned} \Omega_1 &= -\frac{1}{R}z \int_0^\infty d\alpha \left\{ \Theta(q_1 - \alpha) [1 - \Theta(q_2 - \alpha) \Theta(\alpha - q) \Theta(\alpha - k_t)] + \right. \\ &\quad \left. + \Theta(\alpha - q_1) \Theta(q_2 - \alpha) \Theta(\alpha - q) \Theta(k_t - \alpha) \Theta(\kappa_{t,1}^2 q_1^2 - \min[\kappa_{t,1}^2, \kappa_{t,2}^2] q^2) + \right. \\ &\quad \left. - \Theta(q_1 - \alpha) \Theta(\alpha - q_2) \Theta(\alpha - q) \Theta(\alpha - k_t) \Theta(\kappa_{t,2}^2 q_2^2 - \min[\kappa_{t,1}^2, \kappa_{t,2}^2] q^2) \right\} \\ &= -\frac{1}{R}z \left\{ q_1 - (\min[q_1, q_2] - \max[q, k_t]) \Theta(\min[q_1, q_2] - \max[q, k_t]) + \right. \\ &\quad \left. + (\min[q_2, k_t] - \max[q_1, q]) \Theta(\min[q_2, k_t] - \max[q_1, q]) \Theta(q_1^2 - q^2 \min[1, \kappa_{t,2}^2/\kappa_{t,1}^2]) + \right. \\ &\quad \left. - (q_1 - \max[q_2, q, k_t]) \Theta(q_1 - \max[q_2, q, k_t]) \Theta(q_2^2 - q^2 \min[\kappa_{t,1}^2/\kappa_{t,2}^2, 1]) \right\}, \quad (4.18) \end{aligned}$$

where we used the fact that the distances d_{ij} between partons involve the transverse momenta κ with respect to the beam direction which must be distinguished from k_t , the transverse momentum with respect to the jet direction, and we performed the trivial α integral extracting the crucial $1/R$ dependence we seek.

Note that the transverse momenta with respect to the beam are essentially the longitudinal momenta with respect to the jet (recall that we are looking at jet production at ninety degrees to the beam as well as parton emission very close to the triggered hard parton, $\alpha_i \gg \beta_i$), so we can, to our accuracy, just use:

$$\kappa_{t,1} \approx \frac{\sqrt{s}}{2} \alpha_1 = \frac{\sqrt{s}}{2} z\alpha, \quad \kappa_{t,2} \approx \frac{\sqrt{s}}{2} \alpha_2 = \frac{\sqrt{s}}{2} (1-z)\alpha. \quad (4.19)$$

Thus we can express eq. (4.18) as:

$$\begin{aligned} \Omega_1 &= -\frac{1}{R}z \left\{ q_1 - \Psi(\min[q_1, q_2] - \max[q, k_t]) + \right. \\ &\quad \left. + \Psi(\min[q_2, k_t] - \max[q_1, q]) \times \Theta(q_1 - q \min[1, (1-z)/z]) + \right. \\ &\quad \left. - \Psi(q_1 - \max[q_2, q, k_t]) \times \Theta(q_2 - q \min[z/(1-z), 1]) \right\}, \quad (4.20) \end{aligned}$$

where we defined the function $\Psi(x) = x \Theta(x)$. Here $\Psi(x) = x$ if $x > 0$ and $\Psi(x) = 0$ otherwise.

A similar result is obtained for the $\delta p_t(k_2)$ contribution:

$$\begin{aligned} \Omega_2 = & -\frac{1}{R}(1-z) \left\{ q_2 - \Psi(\min[q_1, q_2] - \max[q, k_t]) + \right. \\ & + \Psi(\min[q_1, k_t] - \max[q_2, q]) \times \Theta(q_2 - q \min[1, z/(1-z)]) + \\ & \left. - \Psi(q_2 - \max[q_1, q, k_t]) \times \Theta(q_1 - q \min[(1-z)/z, 1]) \right\}. \end{aligned} \quad (4.21)$$

The subtracted Ω_3 contribution is simply the subtraction of the naïve trigger function computed before:

$$\Omega_3 = -\frac{1}{R} \sqrt{k_t^2 + m^2}. \quad (4.22)$$

The overall result for the non-inclusive trigger function is then given by combining the Ω_1 , Ω_2 and Ω_3 terms and takes the form:

$$\Omega_{ni} = -\frac{1}{R} \left(k_{t1} + k_{t2} - \sqrt{k_t^2 + m^2} + f(q_1, q_2, z) \right), \quad (4.23)$$

which we wrote in terms of the k_{ti} for the sake of easier comparison with the corresponding expression eq. (3.31) in ref. [21]. The first three terms of the result correspond to the non-inclusive trigger function computed for event shapes while the extra term $f(q_1, q_2, z)$ will break the universality of the Milan factor.

To express the result in terms familiar from refs. [20, 21] we introduce the dimensionless ratios u_1 and u_2 , where

$$\vec{u}_i = \frac{\vec{q}_i}{q}. \quad (4.24)$$

Defining the trigger function $\tilde{\Omega}(u_1, u_2)$ so that $\Omega_{ni}(q_1, q_2) = -q/R \tilde{\Omega}(u_1, u_2)$, and adding together the functions Ω_1 , Ω_2 and $-\Omega_3$, we arrive at:

$$\begin{aligned} \tilde{\Omega}(u_1, u_2) = & zu_1 + (1-z)u_2 - \sqrt{zu_1^2 + (1-z)u_2^2} + \\ & - \Psi \left(\min[u_1, u_2] - \max \left[1, \sqrt{zu_1^2 + (1-z)u_2^2} - z(1-z) \right] \right) + \\ & + z \left\{ \Psi \left(\min \left[u_2, \sqrt{zu_1^2 + (1-z)u_2^2} - z(1-z) \right] - \max[u_1, 1] \right) \times \right. \\ & \times \Theta(u_1 - \min[1, (1-z)/z]) + \\ & - \Psi \left(u_1 - \max \left[u_2, 1, \sqrt{zu_1^2 + (1-z)u_2^2} - z(1-z) \right] \right) \times \\ & \left. \times \Theta(u_2 - \min[z/(1-z), 1]) \right\} + \\ & + (1-z) \left\{ \Psi \left(\min \left[u_1, \sqrt{zu_1^2 + (1-z)u_2^2} - z(1-z) \right] - \max[u_2, 1] \right) \times \right. \\ & \times \Theta(u_2 - \min[z/(1-z), 1]) + \\ & - \Psi \left(u_2 - \max \left[u_1, 1, \sqrt{zu_1^2 + (1-z)u_2^2} - z(1-z) \right] \right) \times \\ & \left. \times \Theta(u_1 - \min[1, (1-z)/z]) \right\}, \end{aligned} \quad (4.25)$$

where in finding the above we made use of the relations:

$$\beta = \beta_1 + \beta_2 = \frac{k_t^2 + m^2}{s\alpha}, \quad \beta_1 = \frac{k_{t1}^2}{s\alpha_1} = \frac{z}{s\alpha} q_1^2, \quad \beta_2 = \frac{k_{t2}^2}{s\alpha_2} = \frac{1-z}{s\alpha} q_2^2, \quad m^2 = z(1-z)q^2, \quad (4.26)$$

which imply $(k_t^2 + m^2)/q^2 = zu_1^2 + (1-z)u_2^2$.

Once again the first line of the above result corresponds to the Milan factor for event-shape variables, and is the same trigger function as encountered in refs. [20, 21], while the remainder will constitute the correction to it from k_t clustering.

The result corresponding to the anti- k_t algorithm is also just the first line of the above result (4.25) since there we have:

$$\Omega_1 = -\frac{1}{R}z \int \frac{d\alpha}{\alpha} \alpha \Theta(q_1 - \alpha), \quad (4.27)$$

which gives $\Omega_1 = -q/R(zu_1)$. Similarly $\Omega_2 = -q/R(1-z)u_2$, and the naïve piece Ω_3 is of course common to both algorithms: $\Omega_3 = -q/R\sqrt{zu_1^2 + (1-z)u_2^2}$.

Thus in the anti- k_t algorithm after combining all the pieces we obtain that the result will be identical to that for event-shape variables, computed in refs. [20, 21].

Having derived the trigger function in the k_t algorithm we now proceed to perform the integration over the remaining variables, including also the decay matrix element M^2 as required by eq. (3.20). The details of the integration procedure will be consigned to the appendix and in the next section we shall quote the result and combine it with the naïve and inclusive terms.

5 Results and conclusions

In this section we will mention the result obtained for the non-inclusive correction using the trigger function of the previous section. We shall then combine the result with the naïve and inclusive pieces computed before.

After computing the trigger function Ω_{ni} described in the previous section one inserts the result into eq. (3.20) to obtain the result for $\langle \delta p_t \rangle^{ni}$. In order to do so we need to integrate over the parton decay phase-space including the squared matrix element M^2 describing gluon decay. The details of this integration are mentioned in the appendix and here we just quote the result analogous to eq. (4.8) of [21]:

$$\langle \delta p_t \rangle^{ni} = \langle \delta p_t \rangle^0 r_{ni}, \quad (5.1)$$

where for the k_t algorithm we have a result for r_{ni} different from that for the event-shape variables studied in refs. [20, 21]:

$$\begin{aligned} r_{ni} &= \frac{2}{\beta_0} (-2.145C_A + 0.610C_A - 0.097n_f) \\ &= \frac{1}{\beta_0} (-3.071C_A - 0.194n_f), \end{aligned} \quad (5.2)$$

where the results in the first line correspond to the “soft gluon”, “hard gluon” and “hard quark” contributions in the terminology of refs. [20, 21].

The corresponding result for the anti- k_t algorithm coincides with that obtained for event shapes [20, 21]:

$$r_{ni} = \frac{2}{\beta_0}(-1.227C_A + 0.365C_A - 0.052n_f). \quad (5.3)$$

Combining the result obtained for r_{ni} with that for the inclusive piece (eq. (3.18)) and the naïve result (eq. (3.15)) one obtains the full result for jets defined in the k_t algorithm:

$$\begin{aligned} \langle \delta p_t \rangle &= \frac{-2A_1}{R} (1 + r_{in} + r_{ni}) \\ &= \frac{-2A_1}{R} (1.01), \end{aligned} \quad (5.4)$$

for $n_f = 3$ (for $n_f = 0$ the rescaling factor is 1.06). The factor multiplying the naïve result is thus found to be 1.01 compared to 1.49 for event shapes [20, 21] and the anti- k_t algorithm. Here we took $n_f = 3$ since we are dealing with the non-perturbative region but in the corresponding perturbative estimates one should of course take $n_f = 5$. To obtain the results for a gluon jet one can just replace C_F in A_1 , by C_A .

Thus compared to the thrust $1/Q$ correction which has a Milan factor of 1.49 the $1/R$ term of the jet hadronisation for quark jets in the k_t algorithm has a coefficient that is $0.5 \times 1.01/1.49 \approx 0.34$ times the coefficient of $1/Q$ for e^+e^- thrust instead of 0.5 as observed at the single-gluon level in [14]. For the anti- k_t algorithm the result is still one-half of that for the thrust after inclusion of the Milan factor 1.49 in both cases. The somewhat smaller hadronisation correction predicted here for the k_t algorithm compared to the anti- k_t algorithm was also observed in the Monte-Carlo studies of ref. [14].

To summarise, we began this paper by noting that the single-gluon calculations performed in ref. [14] are inadequate in terms of determining the size of the hadronisation correction to jet p_t relative to the known $1/Q$ corrections for event shapes. The reason for this inadequacy was merely the fact that a two-loop calculation had been shown to be necessary in refs. [20, 21] to unambiguously determine the size of $1/Q$ hadronisation corrections to event shapes.

Our aim in this paper was to carry out a similar calculation for the jet p_t so as to finally put it on the same footing as event shape variables, which makes the comparison feasible. We carried out such a calculation and pointed out that as for event-shape variables one can write the result as a two-loop rescaling factor multiplying the naïve single-gluon result. *Unlike* event-shape variables however, for jets defined in the k_t algorithm the rescaling factor is not the universal Milan factor 1.49 turning out instead to be 1.01. We confirmed the expectation of ref. [3] that for jets defined in the anti- k_t algorithm the rescaling factor is 1.49, as for event-shape variables.

We emphasise the point made earlier, that while in order to make a link with the previous treatment [14] we chose a specific process (jets with the maximum possible p_t at right-angles to the beam), the calculation of the $1/R$ -term is entirely independent of the details of the hard process such as geometry of the hard ensemble. This is because the $1/R$ -term originates from emission and branching collinear to the hard parton that triggers the jet while only inter-jet radiation would be sensitive to the ensemble geometry. Due to

the fact that there is dynamical independence from the details of the hard process as well as kinematical independence,⁸ the $1/R$ -term does not depend for instance on the rapidity of the emitting hard parton.

Physically the issue at hand is that at small R the problem becomes a multi-scale problem involving the hard scale p_t and the smaller but still perturbative scale Rp_t . Thus perturbatively we encounter $\alpha_s \ln R$ terms mentioned in [14] which are clearly independent of the details of the hard process as explained in ref. [14]. In the non-perturbative region we have to deal with the evolution from the scale Rp_t down to the hadronisation scale Λ_{QCD} , which leads to power-suppressed terms Λ_{QCD}/R which are similarly independent of the hard process constituting the non-perturbative tail of the perturbative $\ln R$ pieces. This independence of the $1/R$ -term from the hard process was exploited in ref. [14] where we applied our results to jets in any hard process away from the threshold limit and found good agreement with Monte Carlo event generators. Finite- R corrections were also reported in ref. [14]. These *do depend* on details such as jet rapidities as would their associated Milan factor. We do not envisage including these terms in phenomenological investigations due to the fact that they will be in direct competition with the underlying-event which also has a regular dependence on R .

We also note that we have presented here the calculation of correlated two-parton emission and neglected higher correlations involving three or more partons which would come in beyond our two-loop accuracy. However the two-loop calculation we performed here corresponds to the accuracy needed to fix the scale of the coupling or equivalently to fix a value for Λ_{QCD} , which directly controls the size of the non-perturbative contributions. Hence while one expects the effects of the two-loop calculation to be crucial, higher-loop calculations can be expected to contribute at most corrections of relative order α_s/π , which could change the value of the two-loop factors at most at the twenty-percent level [20]. Interestingly Sterman and Lee [25] have recently shown within the context of soft-collinear effective theory that for observables such as event-shape variables that are linear in emitted parton transverse momenta, one can generally demonstrate universality of power corrections without resorting to fixed-order perturbative calculations beyond the naïve massive gluon results. It would be of interest to revisit their arguments in the context of the observable studied in the current paper.

To conclude we point out that the potential experimental studies mentioned in [14] should of course take into account the result of the calculation performed here for jets defined in the k_t and anti- k_t algorithms. The calculation for jets defined in the SISCone and the Cambridge/Aachen algorithms is work in progress [26].

Acknowledgments

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⁸For the $1/R$ -piece the observable is simply proportional to the energy carried-off by the emitted gluon and we need not invoke the triggered and recoil jet masses mentioned before.

A Numerical result for the non-inclusive correction

In this section we evaluate the integral (3.20) relevant to the non-inclusive correction.

A.1 Two-loop matrix element and phase-space

To proceed we shall need the expression for the decay matrix element squared. In terms of the variables z , u_1 and u_2 this can be written as [20]:

$$M^2 = \frac{1}{m^2 q^2} \frac{\mathcal{M}^2}{zu_1^2 + (1-z)u_2^2}, \quad (\text{A.1})$$

with

$$\mathcal{M}^2 = 2C_A(2\mathcal{S} + \mathcal{H}_g) + 2n_f \mathcal{H}_q. \quad (\text{A.2})$$

The components of \mathcal{M}^2 are functions of u_1 , u_2 and z alone:

$$2\mathcal{S} = \frac{zu_1^2 + (1-z)u_2^2}{z(1-z)} \left(\frac{1}{u_1^2} + \frac{1}{u_2^2} - \frac{1}{u_1^2 u_2^2} \right), \quad (\text{A.3})$$

$$\mathcal{H}_g = -2 + z(1-z) \frac{(u_1^2 - u_2^2)^2}{zu_1^2 + (1-z)u_2^2} - \frac{u_1^2 + u_2^2}{2} \left(\frac{1}{u_1^2} + \frac{1}{u_2^2} - \frac{1}{u_1^2 u_2^2} \right), \quad (\text{A.4})$$

$$\mathcal{H}_q = 1 - z(1-z) \frac{(u_1^2 - u_2^2)^2}{zu_1^2 + (1-z)u_2^2}, \quad (\text{A.5})$$

which successively represent the soft gluon, hard gluon and hard quark contributions. We shall also need to express the decay phase-space which appears in the curly brackets of eq. (3.20) in terms of the same variables. We have [20]:

$$\frac{d\phi}{2\pi} dz dk_t^2 = dz \frac{4q^2}{\pi} \frac{u_1 du_1 u_2 du_2}{\sqrt{J}}, \quad (\text{A.6})$$

with $J = [(u_1 + u_2)^2 - 1][1 - (u_1 - u_2)^2]$.

Substituting the decay matrix element and phase-space into eq. (3.20) we notice⁹ that the term in the curly brackets of the same equation is in fact proportional¹⁰ to m , thus we write it as:

$$\frac{1}{\beta_0} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^1 dz \int dk_t^2 m^2 \frac{M^2}{2!} \Omega_{ni} = -\frac{1}{R} 2m r_{ni}, \quad (\text{A.7})$$

where r_{ni} is a constant factor to be evaluated. The remaining integral in eq. (3.20) over the mass m is trivial and we can thus relate the result to $\langle \delta p_t \rangle^0$ via:

$$\langle \delta p_t \rangle^{ni} = \langle \delta p_t \rangle^0 r_{ni}. \quad (\text{A.8})$$

In the following subsections we evaluate the constant r_{ni} .

⁹Recall that $\Omega_{ni} = -q/R \tilde{\Omega}(u_1, u_2, z)$, with $q = m/\sqrt{z(1-z)}$.

¹⁰Strictly speaking this is only true if we set the upper limit of the k_t integral to infinity. Formally the upper limit is Q but setting it to ∞ generates subleading corrections $\mathcal{O}(m^2/Q)$ which we ignore.

A.2 The non-inclusive rescaling factor r_{ni}

From eq. (A.7) we have:

$$r_{ni} = -\frac{R}{m} \frac{1}{2\beta_0} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^1 dz \int dk_t^2 m^2 \frac{M^2}{2!} \Omega_{ni}, \quad (\text{A.9})$$

where as before we write $\Omega_{ni} = -q/R \tilde{\Omega}(u_1, u_2)$. Thus using the results from the previous subsection we can express this integral in terms of the variables u_1, u_2 and z as:

$$r_{ni} = \frac{1}{\pi\beta_0} \int dz \frac{u_1 du_1 u_2 du_2}{\sqrt{J}} \frac{M^2}{z u_1^2 + (1-z) u_2^2} \frac{1}{\sqrt{z(1-z)}} \tilde{\Omega}(u_1, u_2), \quad (\text{A.10})$$

where we used $q = m/\sqrt{z(1-z)}$. The limits of integration are $0 < z < 1$, $u_1 > 0$ and $u_2 > 0$. Notice also that the ‘‘Jacobian factor’’ J is positive which also restricts u_1 and u_2 such that $u_1 + u_2 > 1$ and $-1 < u_1 - u_2 < 1$. We thus define $u_{\pm} = u_1 \pm u_2$ and express the integral in terms of u_+ and u_- :

$$r_{ni} = \frac{1}{\pi\beta_0} \int_0^1 \frac{dz}{\sqrt{z(1-z)}} \frac{1}{2} \int_{-1}^1 \frac{du_-}{\sqrt{1-u_-^2}} \int_1^{\infty} \frac{du_+}{\sqrt{u_+^2-1}} u_1 u_2 \frac{M^2}{z u_1^2 + (1-z) u_2^2} \tilde{\Omega}(u_1, u_2), \quad (\text{A.11})$$

where $u_+ > 1$ and $-1 < u_- < 1$.

By symmetry we just consider the region of the phase-space of integration defined by $0 < u_- < 1$ (i.e. $u_1 > u_2$) and multiply the result at the end by a factor 2. This way the trigger function (eq. (4.25)) takes the simpler form ($u_1 > u_2$):

$$\begin{aligned} \tilde{\Omega}(u_1, u_2) = & z u_1 + (1-z) u_2 - \sqrt{z u_1^2 + (1-z) u_2^2} + \\ & - \Psi \left(u_2 - \max \left[1, \sqrt{z u_1^2 + (1-z) u_2^2 - z(1-z)} \right] \right) + \\ & - z \Psi \left(u_1 - \max \left[u_2, 1, \sqrt{z u_1^2 + (1-z) u_2^2 - z(1-z)} \right] \right) \Theta(u_2 - \min[z/(1-z), 1]) + \\ & + (1-z) \Psi \left(\sqrt{z u_1^2 + (1-z) u_2^2 - z(1-z)} - \max[u_2, 1] \right) \Theta(u_2 - \min[z/(1-z), 1]), \end{aligned} \quad (\text{A.12})$$

where we made use of the easily-derived inequality $u_2 < \sqrt{z u_1^2 + (1-z) u_2^2} < u_1$, for $0 < z < 1$, and hence $\sqrt{z u_1^2 + (1-z) u_2^2 - z(1-z)} < u_1$.

To further simplify the trigger function one needs to consider in detail the relative size of the various components of the functions Ψ above. To this end we proceed in the following way.

A.3 Simplification of the trigger function

We consider all possible permutations of the relative size of $\sqrt{z u_1^2 + (1-z) u_2^2 - z(1-z)}$, u_1 , u_2 , and 1. Thus we discuss the following scenarios:

- $1 < u_2 < \sqrt{z u_1^2 + (1-z) u_2^2 - z(1-z)} < u_1$ or $1 < \sqrt{z u_1^2 + (1-z) u_2^2 - z(1-z)} < u_2 < u_1$. We call the corresponding region of the phase-space ‘‘A’’, thus:

$$\tilde{\Omega}_A(u_1, u_2) = \sqrt{z u_1^2 + (1-z) u_2^2 - z(1-z)} - \sqrt{z u_1^2 + (1-z) u_2^2}. \quad (\text{A.13})$$

- $\sqrt{zu_1^2 + (1-z)u_2^2 - z(1-z)} < 1 < u_2 < u_1$. We label the corresponding region “ \tilde{A} ”. Here we have:

$$\tilde{\Omega}_{\tilde{A}}(u_1, u_2) = 1 - \sqrt{zu_1^2 + (1-z)u_2^2}. \quad (\text{A.14})$$

- $u_2 < 1 < \sqrt{zu_1^2 + (1-z)u_2^2 - z(1-z)} < u_1$. We name this region “ B ”. In this case we have:

$$\begin{aligned} \tilde{\Omega}_B(u_1, u_2) &= zu_1 + (1-z)u_2 - \sqrt{zu_1^2 + (1-z)u_2^2} + \\ &+ \left[\sqrt{zu_1^2 + (1-z)u_2^2 - z(1-z)} - zu_1 - (1-z) \right] \Theta(u_2 - z/(1-z)) \Theta(1-2z). \end{aligned} \quad (\text{A.15})$$

- $u_2 < \sqrt{zu_1^2 + (1-z)u_2^2 - z(1-z)} < 1 < u_1$ or $\sqrt{zu_1^2 + (1-z)u_2^2 - z(1-z)} < u_2 < 1 < u_1$. We call this region “ C ”, hence:

$$\begin{aligned} \tilde{\Omega}_C(u_1, u_2) &= zu_1 + (1-z)u_2 - \sqrt{zu_1^2 + (1-z)u_2^2} + \\ &- z(u_1 - 1) \Theta(u_2 - z/(1-z)) \Theta(1-2z). \end{aligned} \quad (\text{A.16})$$

- $u_2 < \sqrt{zu_1^2 + (1-z)u_2^2 - z(1-z)} < u_1 < 1$ or $\sqrt{zu_1^2 + (1-z)u_2^2 - z(1-z)} < u_2 < u_1 < 1$. We name this region “ D ”, where:

$$\tilde{\Omega}_D(u_1, u_2) = zu_1 + (1-z)u_2 - \sqrt{zu_1^2 + (1-z)u_2^2}. \quad (\text{A.17})$$

We summarise the above results for the trigger function in table 2, for which we define:

$$\Omega_m = zu_1 + (1-z)u_2 - \sqrt{zu_1^2 + (1-z)u_2^2}, \quad (\text{A.18})$$

which is just the non-inclusive trigger function for event shapes and the anti- k_t algorithm. We also show in figure 1 the (u_2, u_1) phase-space and the relevant regions of integration. We note that the overall region of integration is $u_+ > 1$ and $-1 < u_- < 1$ [20], but here we are only considering the region $0 < u_- < 1$ as we stated before.

A.4 Convergence issue

Having simplified the trigger function we now check the convergence of the integral (A.11).

First of all we discuss the limit $u_+ \rightarrow \infty$. Other than the trigger function, the integrand in eq. (A.11) behaves as $1/u_+$ for large u_+ . The trigger function in this limit (corresponding to region A) is given by:

$$\begin{aligned} \tilde{\Omega}_A &= \frac{\sqrt{zu_1^2 + (1-z)u_2^2 - z(1-z)} - \sqrt{zu_1^2 + (1-z)u_2^2}}{2z(1-z)} \\ &= \frac{-\sqrt{(u_+ + (2z-1)u_-)^2 + 4z(1-z)(u_-^2 - 1)} + \sqrt{(u_+ + (2z-1)u_-)^2 + 4z(1-z)u_-^2}}{2z(1-z)} \\ &\xrightarrow{u_+ \rightarrow \infty} -\frac{z(1-z)}{u_+}, \end{aligned} \quad (\text{A.19})$$

which clearly vanishes as $u_+ \rightarrow \infty$, thus ensuring convergence [20].

Region	$\tilde{\Omega}(u_1, u_2)$
A	$\sqrt{zu_1^2 + (1-z)u_2^2 - z(1-z)} - \sqrt{zu_1^2 + (1-z)u_2^2}$ $= \Omega_m + \sqrt{zu_1^2 + (1-z)u_2^2 - z(1-z)} - zu_1 - (1-z)u_2$
\tilde{A}	$1 - \sqrt{zu_1^2 + (1-z)u_2^2} = \Omega_m + 1 - zu_1 - (1-z)u_2$
B	$\Omega_m + \left[\sqrt{zu_1^2 + (1-z)u_2^2 - z(1-z)} - zu_1 - (1-z) \right] \times$ $\times \Theta[u_2 - z/(1-z)]\Theta(1/2 - z)$
C	$\Omega_m - z(u_1 - 1)\Theta[u_2 - z/(1-z)]\Theta(1/2 - z)$
D	Ω_m

Table 2. The trigger function in different regions of the phase-space. See figure 1.

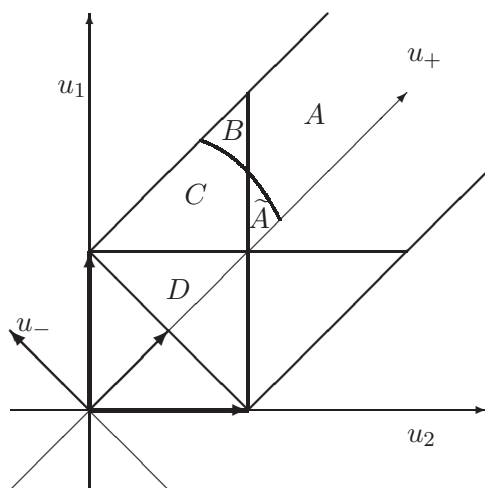


Figure 1. The (u_2, u_1) phase-space of integration for eq. (A.11). See table 2 for the trigger function $\tilde{\Omega}$ in various regions. Here the ellipse (curved line) defines the boundary between regions governed by the relative size of $zu_1^2 + (1-z)u_2^2 - z(1-z)$ and 1. This ellipse ranges from being a straight line parallel to u_1 axis (when $z = 0$) or u_2 axis (when $z = 1$) to a circle of radius $\sqrt{5/2}$ (when $z = 1/2$).

Next we discuss the limit $z(1-z) \rightarrow 0$ and verify the convergence in various regions of the (u_2, u_1) phase-space. Dealing first with region A we notice that when $z(1-z) \rightarrow 0$ the trigger function has the behaviour:

$$\tilde{\Omega}_A \xrightarrow{z(1-z) \rightarrow 0} -\frac{z(1-z)}{u_+ + (2z-1)u_-}, \tag{A.20}$$

which removes the singularity in the matrix element of the form $1/z(1-z)$ (see eq. (A.3)). We do not here encounter the problem of the edge of the phase-space as in [20] ($u_+ \rightarrow 1$ and $u_- \rightarrow \pm 1$) since we are away from these corners of the phase-space in region A, i.e.

the denominator in eq. (A.20) never approaches zero.

In region \tilde{A} the trigger function has the following form:

$$\tilde{\Omega}_{\tilde{A}} = 1 - \sqrt{zu_1^2 + (1-z)u_2^2}. \quad (\text{A.21})$$

Here we have the condition $zu_1^2 + (1-z)u_2^2 - z(1-z) < 1$, which can be written as $-\tilde{\Omega}_{\tilde{A}} < z(1-z)/2 + \mathcal{O}(z^2(1-z)^2)$ when $z(1-z) \rightarrow 0$. Thus the trigger function here removes the singularity in the matrix element of the form $1/z(1-z)$.

Next we consider region B . The convergence of the integral involving Ω_m has been discussed in [20], so we only check the convergence of the integral involving the ‘‘correction’’ term $\tilde{\Omega}_B^{\text{corr}} = \sqrt{zu_1^2 + (1-z)u_2^2 - z(1-z)} - zu_1 - (1-z)$, which is present if $z < 1/2$ (i.e. we only worry about $z \rightarrow 0$ singularity). In this region we have the condition $zu_1^2 + (1-z)u_2^2 - z(1-z) > 1$, which can be written as $-\tilde{\Omega}_B^{\text{corr}} < (u_1 - 1)z$, where $0 < u_1 - 1 < 1$. Thus we deduce that the trigger function removes the singularity $1/z$ in the matrix element when $z \rightarrow 0$.

Similarly for region C the correction term (which is present if $z < 1/2$) is: $-z(u_1 - 1)$, which clearly removes the singularity $1/z$ in the matrix element since $0 < u_1 - 1 < 1$.

Lastly the integral in region D is obviously convergent since the trigger function there is equal to Ω_m . Hence we deduce that the trigger function always removes the $1/z(1-z)$ singularity in the matrix element.

We note that here we do not specifically have the problem of the edge of the phase-space which appears in the Milan factor computation [20] (except for Ω_m , which has been discussed in the same paper).

A.5 Numerical result

Inserting the results for the trigger function from table 2 into eq. (A.11), taking into account the appropriate phase-space, we are able to numerically compute the result for the non-inclusive factor r_{ni} . We present here the numerical result for each region of the phase-space.

We first compute the result for integration over Ω_m which appears in all the regions (see table 2). This has already been done [20] and the result is:

$$r_{ni}^m = \frac{1}{\beta_0}(-1.227C_A + 0.365C_A - 0.052n_f), \quad (\text{A.22})$$

where after multiplying by the factor 2 (accounting for the fact that we have considered just $u_- > 0$) we arrive at the result in [20]. Next we compute the ‘‘corrections’’ region-by-region.

Region D has no corrections, so we begin with region C : the correction term is $-z(u_1 - 1)$, and the phase-space is $\Theta[u_2 - z/(1-z)]\Theta(1/2 - z)\Theta(1 - zu_1^2 - (1-z)u_2^2 + z(1-z))\Theta(u_-^2 - (u_+ - 2)^2)$, with $0 < u_- < 1$ and $1 < u_+ < 3$. The numerical result in this case is:

$$r_{ni}^C = \frac{1}{\beta_0}(-0.262C_A + 0.072C_A - 0.008n_f). \quad (\text{A.23})$$

The correction term for region B is $\sqrt{zu_1^2 + (1-z)u_2^2 - z(1-z)} - zu_1 - (1-z)$, and the phase-space is $\Theta[u_2 - z/(1-z)]\Theta(1/2 - z)\Theta(-1 + zu_1^2 + (1-z)u_2^2 - z(1-z))\Theta(u_-^2 - (u_+ - 2)^2)$,

Region	anti- k_t	k_t
A	$-0.557C_A + 0.123C_A - 0.016n_f$	$-1.167C_A + 0.277C_A - 0.051n_f$
\tilde{A}	$-0.000C_A + 0.000C_A - 0.000n_f$	$-0.003C_A + 0.001C_A - 0.000n_f$
B	$-0.245C_A + 0.039C_A - 0.004n_f$	$-0.288C_A + 0.056C_A - 0.006n_f$
C	$-0.982C_A + 0.310C_A - 0.017n_f$	$-1.244C_A + 0.382C_A - 0.025n_f$
D	$+0.557C_A - 0.107C_A - 0.015n_f$	$+0.557C_A - 0.107C_A - 0.015n_f$
Sum	$-1.227C_A + 0.365C_A - 0.052n_f$	$-2.145C_A + 0.610C_A - 0.097n_f$

Table 3. Comparison between the non-inclusive coefficient r_{ni} in the k_t and anti- k_t algorithms.

with $0 < u_- < 1$ and $1 < u_+ < 3$. The numerical result is:

$$r_{ni}^B = \frac{1}{\beta_0}(-0.043C_A + 0.017C_A - 0.002n_f). \quad (\text{A.24})$$

For region \tilde{A} the correction is $1 - zu_1 - (1 - z)u_2$ and the phase-space is $\Theta((u_+ - 2)^2 - u_-^2)\Theta(1 - zu_1^2 - (1 - z)u_2^2 + z(1 - z))$, with $2 < u_+ < 1 + \sqrt{2}$ and $0 < u_- < 1$. The numerical result is:

$$r_{ni}^{\tilde{A}} = \frac{1}{\beta_0}(-0.003C_A + 0.001C_A - 0.000n_f). \quad (\text{A.25})$$

Finally we treat region A . Here the correction reads: $\sqrt{zu_1^2 + (1 - z)u_2^2 - z(1 - z)} - zu_1 - (1 - z)u_2$, with the phase-space being $\Theta((u_+ - 2)^2 - u_-^2)\Theta(-1 + zu_1^2 + (1 - z)u_2^2 - z(1 - z))$, and $2 < u_+ < \infty$ and $0 < u_- < 1$. The numerical result is:

$$r_{ni}^A = \frac{1}{\beta_0}(-0.610C_A + 0.154C_A - 0.035n_f). \quad (\text{A.26})$$

Thus the final result for the non-inclusive correction factor r_{ni} , which is $r_{ni}^m + r_{ni}^A + r_{ni}^{\tilde{A}} + r_{ni}^B + r_{ni}^C$, multiplied by a factor 2, accounting for the fact that we have just considered $u_- > 0$, is:

$$\begin{aligned} r_{ni} &= \frac{2}{\beta_0}(-2.145C_A + 0.610C_A - 0.097n_f) \\ &= \frac{1}{\beta_0}(-3.071C_A - 0.194n_f), \end{aligned} \quad (\text{A.27})$$

where we have written the result in all the above as a sum of three terms to show the contributions from the soft gluon, hard gluon and hard quark matrix elements respectively as in [20].

Finally we note that $\beta_0 = 11/3C_A - 2/3n_f = 9(11)$ for $n_f = 3(0)$.

A.6 Comparison between k_t and anti- k_t results

We present in table 3 a comparison between the r_{ni} results for the k_t and anti- k_t algorithms in various regions of the phase-space. A factor $2/\beta_0$ is left out.

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